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Research Article

Some Comparison Inequalities for Generalized Muirhead and Identric Means

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For $x, y > 0$, $a, b \in \mathbb{R}$, with $a + b \neq 0$, the generalized Muirhead mean $M(a, b; x, y)$ with parameters a and b and the identric mean $I(x, y)$ are defined by $M(a, b; x, y) = ((x^a y^b + x^b y^a)/2)^{1/(a+b)}$ and $I(x, y) = (1/e)(y^y/x^x)^{1/(y-x)}$, $x \neq y$, $I(x, y) = x$, $x = y$, respectively. In this paper, the following results are established: (1) $M(a, b; x, y) > I(x, y)$ for all $x, y > 0$ with $x \neq y$ and $(a, b) \in \{(a, b) \in \mathbb{R}^2 : a + b > 0, ab \leq 0, 2(a-b)^2 - 3(a+b) + 1 \geq 0, 3(a-b)^2 - 2(a+b) \geq 0\}$; (2) $M(a, b; x, y) < I(x, y)$ for all $x, y > 0$ with $x \neq y$ and $(a, b) \in \{(a, b) \in \mathbb{R}^2 : a \geq 0, b \geq 0, 3(a-b)^2 - 2(a+b) \leq 0\} \cup \{(a, b) \in \mathbb{R}^2 : a + b < 0\}$; (3) if $(a, b) \in \{(a, b) \in \mathbb{R}^2 : a > 0, b > 0, 3(a-b)^2 - 2(a+b) > 0\} \cup \{(a, b) \in \mathbb{R}^2 : ab < 0, 3(a-b)^2 - 2(a+b) < 0\}$, then there exist $x_1, y_1, x_2, y_2 > 0$ such that $M(a, b; x_1, y_1) > I(x_1, y_1)$ and $M(a, b; x_2, y_2) < I(x_2, y_2)$.

1. Introduction

For $x, y > 0$, $a, b \in \mathbb{R}$, with $a + b \neq 0$, the generalized Muirhead mean $M(a, b; x, y)$ with parameters a and b and the identric mean $I(x, y)$ are defined by

$$M(a, b; x, y) = \left(\frac{x^a y^b + x^b y^a}{2} \right)^{1/(a+b)}, \quad (1.1)$$

$$I(x, y) = \begin{cases} \frac{1}{e} \left(\frac{y^y}{x^x} \right)^{1/(y-x)}, & x \neq y, \\ x, & x = y, \end{cases} \quad (1.2)$$

respectively.

The generalized Muirhead mean was introduced by Trif [1], the monotonicity of $M(a, b; x, y)$ with respect to a or b was discussed, and a comparison theorem and a Minkowski-type inequality involving the generalized Muirhead mean $M(a, b; x, y)$ were discussed.

It is easy to see that the generalized Muirhead mean $M(a, b; x, y)$ is continuous on the domain $\{(a, b; x, y) : a + b \neq 0; x, y > 0\}$ and differentiable with respect to $(x, y) \in (0, +\infty) \times (0, +\infty)$ for fixed $a, b \in \mathbb{R}$ with $a + b \neq 0$. It is symmetric in a and b and in x and y . Many means are special cases of the generalized Muirhead mean, for example,

$$\begin{aligned} M(p, 0; x, y) & \text{ is the power or Hölder mean,} \\ M(0, 1; x, y) & \text{ is the arithmetic mean,} \\ M(a, a; x, y) & \text{ is the geometric mean,} \\ M(0, -1; x, y) & \text{ is the harmonic mean.} \end{aligned} \quad (1.3)$$

The well-known Muirhead inequality [2] implies that if $x, y > 0$ are fixed, then $M(a, b; x, y)$ is Schur convex on the domain $\{(a, b) \in \mathbb{R}^2 : a + b > 0\}$ and Schur concave on the domain $\{(a, b) \in \mathbb{R}^2 : a + b < 0\}$. Chu and Xia [3] discussed the Schur convexity and Schur concavity of $M(a, b; x, y)$ with respect to $(x, y) \in (0, \infty) \times (0, \infty)$ for fixed $a, b \in \mathbb{R}$ with $a + b \neq 0$.

Recently, the identric mean $I(x, y)$ has been the subject of intensive research. In particular, many remarkable inequalities for the identric mean $I(x, y)$ can be found in the literature [4–13].

The power mean of order r of the positive real numbers x and y is defined by

$$M_r(x, y) = \begin{cases} \left(\frac{x^r + y^r}{2} \right)^{1/r}, & r \neq 0, \\ \sqrt{xy}, & r = 0. \end{cases} \quad (1.4)$$

The main properties of the power mean $M_r(x, y)$ are given in [14]. In particular, $M_r(x, y)$ is continuous and increasing with respect to $r \in \mathbb{R}$ for fixed $x, y > 0$. Let $A(x, y) = (1/2)(x + y)$,

$$L(x, y) = \begin{cases} \frac{y - x}{\log y - \log x}, & x \neq y, \\ x, & x = y, \end{cases} \quad (1.5)$$

$G(x, y) = \sqrt{xy}$, and $H(x, y) = 2xy/(x + y)$ be the arithmetic, logarithmic, geometric, and harmonic means of two positive numbers x and y . Then it is well known that

$$\begin{aligned} \min\{x, y\} & < H(x, y) = M(0, -1; x, y) = M_{-1}(x, y) \\ & < G(x, y) = M(a, a; x, y) = M_0(x, y) < L(x, y) < I(x, y) \\ & < A(x, y) = M(0, 1; x, y) = M_1(x, y) < \max\{x, y\} \end{aligned} \quad (1.6)$$

for all $x, y > 0$ with $x \neq y$.

The following sharp inequality is due to Carlson [15]:

$$L(x, y) < \frac{1}{3}M(0, 1; x, y) + \frac{2}{3}M(a, a; x, y) \quad (1.7)$$

for all $x, y > 0$ with $x \neq y$.

Pittenger [16] proved that

$$M\left(\frac{2}{3}, 0; x, y\right) = M_{2/3}(x, y) < I(x, y) < M_{\log 2}(x, y) = M(\log 2, 0; x, y) \quad (1.8)$$

for all $x, y > 0$ with $x \neq y$, and $M_{\log 2}(x, y)$ and $M_{2/3}(x, y)$ are the optimal upper and lower power mean bounds for the identric mean $I(x, y)$.

In [8, 9], Sándor established that

$$I(x, y) > \frac{2}{3}M(0, 1; x, y) + \frac{1}{3}M(a, a; x, y) \quad (1.9)$$

for all $x, y > 0$ with $x \neq y$.

Alzer and Qiu [5] proved the inequalities

$$\alpha M(0, 1; x, y) + (1 - \alpha)M(a, a; x, y) < I(x, y) < \beta M(0, 1; x, y) + (1 - \beta)M(a, a; x, y) \quad (1.10)$$

for all $x, y > 0$ with $x \neq y$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e$.

In [3], Chu and Xia proved that

$$M(a, b; x, y) \geq A(x, y) \quad (1.11)$$

for all $x, y > 0$ and $(a, b) \in \{(a, b) \in \mathbb{R}^2 : (a - b)^2 \geq a + b >, ab \leq 0\}$, and

$$M(a, b; x, y) \leq A(x, y) \quad (1.12)$$

for all $x, y > 0$ and $(a, b) \in \{(a, b) \in \mathbb{R}^2 : (a - b)^2 \leq a + b, a^2 + b^2 \neq 0\} \cup \{(a, b) \in \mathbb{R}^2 : a + b < 0\}$.

Our purpose in what follows is to compare the generalized Muirhead mean $M(a, b; x, y)$ with the identric mean $I(x, y)$. Our main result is Theorem 1.1 which follows.

Theorem 1.1. Suppose that $E_1 = \{(a, b) \in \mathbb{R}^2 : a + b > 0, ab \leq 0, 2(a - b)^2 - 3(a + b) + 1 \geq 0, 3(a - b)^2 - 2(a + b) \geq 0\}$, $E_2 = \{(a, b) \in \mathbb{R}^2 : a \geq 0, b \geq 0, a^2 + b^2 \neq 0, 3(a - b)^2 - 2(a + b) \leq 0\} \cup \{(a, b) \in \mathbb{R}^2 : a + b < 0\}$, and $E_3 = \{(a, b) \in \mathbb{R}^2 : a > 0, b > 0, 3(a - b)^2 - 2(a + b) > 0\} \cup \{(a, b) \in \mathbb{R}^2 : ab < 0, 3(a - b)^2 - 2(a + b) < 0\}$. The following statements hold,

- (1) If $(a, b) \in E_1$, then $M(a, b; x, y) > I(x, y)$ for all $x, y > 0$ with $x \neq y$.
- (2) If $(a, b) \in E_2$, then $M(a, b; x, y) < I(x, y)$ for all $x, y > 0$ with $x \neq y$.
- (3) If $(a, b) \in E_3$, then there exist $x_1, y_1, x_2, y_2 > 0$ such that $M(a, b; x_1, y_1) > I(x_1, y_1)$ and $M(a, b; x_2, y_2) < I(x_2, y_2)$.

2. Lemma

In order to prove Theorem 1.1 we need Lemma 2.1 that follows.

Lemma 2.1. *Let a and b be two real numbers such that $a > b$ and $a+b \neq 0$. Let one define the function $f : [1, +\infty) \rightarrow \mathbb{R}$ as follows:*

$$f(t) = \frac{1}{a+b} \left[-bt^{a-b+1} + at^{a-b} - at^{b-a+1} + bt^{b-a} + (a^2 + b^2 - 2ab - a - b)(t-1) \right], \quad (2.1)$$

then the following statements hold.

- (1) If $b > 0$ and $3(a-b)^2 - 2(a+b) \leq 0$, then $f(t) < 0$ for $t > 1$.
- (2) If $b < 0$, $a+b > 0$, $2(a-b)^2 - 3(a+b) + 1 \geq 0$, and $3(a-b)^2 - 2(a+b) \geq 0$, then $f(t) > 0$ for $t > 1$.
- (3) If $a+b < 0$, then $f(t) < 0$ for $t > 1$.

Proof. Simple computations lead to

$$f(1) = 0, \quad (2.2)$$

$$f'(t) = \frac{1}{a+b} \left[-b(a-b+1)t^{a-b} + a(a-b)t^{a-b-1} + a(a-b-1)t^{b-a} - b(a-b)t^{b-a-1} + a^2 + b^2 - 2ab - a - b \right], \quad (2.3)$$

$$f'(1) = \frac{3(a-b)^2 - 2(a+b)}{a+b}, \quad (2.4)$$

$$f''(t) = (a-b)t^{b-a-2}f_1(t), \quad (2.5)$$

where

$$f_1(t) = \frac{1}{a+b} \left[-b(a-b+1)t^{2a-2b+1} + a(a-b-1)t^{2a-2b} - a(a-b-1)t + b(a-b+1) \right], \quad (2.6)$$

$$f_1(1) = 0, \quad (2.7)$$

$$f'_1(t) = \frac{1}{a+b} \left[-b(a-b+1)(2a-2b+1)t^{2a-2b} + 2a(a-b)(a-b-1)t^{2a-2b-1} - a(a-b-1) \right], \quad (2.8)$$

$$f'_1(1) = \frac{a-b}{a+b} \left[2(a-b)^2 - 3(a+b) + 1 \right], \quad (2.9)$$

$$f''_1(t) = 2(a-b)t^{2a-2b-2}f_2(t), \quad (2.10)$$

where

$$f_2(t) = \frac{1}{a+b}[-b(a-b+1)(2a-2b+1)t + a(a-b-1)(2a-2b-1)], \quad (2.11)$$

$$f_2(1) = \frac{a-b}{a+b} \left[2(a-b)^2 - 3(a+b) + 1 \right], \quad (2.12)$$

$$f'_2(t) = -\frac{b(a-b+1)(2a-2b+1)}{a+b}. \quad (2.13)$$

(1) We divide the proof of Lemma 2.1(1) into two cases.

Case 1. $b > 0$, $3(a-b)^2 - 2(a+b) \leq 0$, and $2(a-b)^2 - 3(a+b) + 1 \leq 0$. From (2.13), (2.12), (2.9), and (2.4), we clearly see that

$$\begin{aligned} f'_2(t) &< 0, & f_2(1) &\leq 0, \\ f'_1(1) &\leq 0, & f'(1) &\leq 0. \end{aligned} \quad (2.14)$$

Therefore, $f(t) < 0$ for $t \in (1, +\infty)$ easily follows from (2.2), (2.5), (2.7), (2.10), and (2.14).

Case 2. $b > 0$, $3(a-b)^2 - 2(a+b) \leq 0$, and $2(a-b)^2 - 3(a+b) + 1 > 0$; we conclude that

$$a < \frac{1}{2}. \quad (2.15)$$

In fact, we clearly see that $2(a-b)^2 - 3(a+b) + 1 = (2a^2 - 3a + 1) - (4ab - 2b^2 + 3b) < 2a^2 - 3a + 1 = (2a-1)(a-1) \leq 0$ for $1/2 \leq a < 1$, and $2(a-b)^2 - 3(a+b) + 1 \leq -(5/3)(a+b) + 1 < -2/3 < 0$ for $a \geq 1$ and $3(a-b)^2 - 2(a+b) \leq 0$.

Equation (2.15) and $3(a-b)^2 - 2(a+b) \leq 0$ imply that

$$\begin{aligned} 2a - 2b - 1 &< 0, \\ a^2 + b^2 - 2ab - a - b &= (a-b)^2 - (a+b) < 0. \end{aligned} \quad (2.16)$$

Therefore, $f(t) < 0$ for $t > 1$ follows from (2.16) together with that $f(t)$ can be rewritten as

$$\begin{aligned} f(t) &= \frac{1}{a+b} \left[at^{b-a+1} \left(t^{2a-2b-1} - 1 \right) - bt^{b-a} \left(t^{2a-2b+1} - 1 \right) \right. \\ &\quad \left. + \left(a^2 + b^2 - 2ab - a - b \right) (t-1) \right]. \end{aligned} \quad (2.17)$$

(2) If $b < 0$, $a + b > 0$, $2(a - b)^2 - 3(a + b) + 1 \geq 0$ and $3(a - b)^2 - 2(a + b) \geq 0$, then from (2.13), (2.12), (2.9), and (2.4) we get

$$\begin{aligned} f_2'(t) &> 0, & f_2(1) &\geq 0, \\ f_1'(1) &\geq 0, & f'(1) &\geq 0. \end{aligned} \quad (2.18)$$

Therefore, $f(t) > 0$ for $t \in (1, +\infty)$ easily follows from (2.2), (2.5), (2.7), and (2.10) together with (2.18).

(3) If $a + b < 0$, then we clearly see that inequalities (2.14) again hold, and $f(t) < 0$ for $t > 1$ follows from (2.2), (2.5), (2.7), and (2.10) together with (2.14). \square

3. Proof of Theorem 1.1

Proof of Theorem 1.1. For convenience, we introduce the following classified regions in \mathbb{R}^2 :

$$\begin{aligned} E_{11} &= \left\{ (a, b) \in \mathbb{R}^2 : a + b > 0, a > 0, b < 0, 2(a - b)^2 - 3(a + b) + 1 \geq 0, \right. \\ &\quad \left. 3(a - b)^2 - 2(a + b) \geq 0 \right\}, \\ E_{12} &= \left\{ (a, b) \in \mathbb{R}^2 : a + b > 0, a < 0, b > 0, 2(a - b)^2 - 3(a + b) + 1 \geq 0, \right. \\ &\quad \left. 3(a - b)^2 - 2(a + b) \geq 0 \right\}, \\ E_{13} &= \left\{ (a, b) \in \mathbb{R}^2 : a = 0, b \geq 1 \right\}, \\ E_{14} &= \left\{ (a, b) \in \mathbb{R}^2 : b = 0, a \geq 1 \right\}, \\ E_{21} &= \left\{ (a, b) \in \mathbb{R}^2 : a > b > 0, 3(a - b)^2 - 2(a + b) \leq 0 \right\}, \\ E_{22} &= \left\{ (a, b) \in \mathbb{R}^2 : b > a > 0, 3(a - b)^2 - 2(a + b) \leq 0 \right\}, \\ E_{23} &= \left\{ (a, b) \in \mathbb{R}^2 : a = 0, 0 < b \leq \frac{2}{3} \right\}, \\ E_{24} &= \left\{ (a, b) \in \mathbb{R}^2 : b = 0, 0 < a \leq \frac{2}{3} \right\}, \\ E_{25} &= \left\{ (a, b) \in \mathbb{R}^2 : a > b, a + b < 0 \right\}, \\ E_{26} &= \left\{ (a, b) \in \mathbb{R}^2 : b > a, a + b < 0 \right\}, \\ E_{27} &= \left\{ (a, b) \in \mathbb{R}^2 : a = b \neq 0 \right\}, \end{aligned}$$

$$\begin{aligned}
E_{31} &= \left\{ (a, b) \in \mathbb{R}^2 : a > b > 0, 3(a-b)^2 - 2(a+b) > 0 \right\}, \\
E_{32} &= \left\{ (a, b) \in \mathbb{R}^2 : b > a > 0, 3(a-b)^2 - 2(a+b) > 0 \right\}, \\
E_{33} &= \left\{ (a, b) \in \mathbb{R}^2 : a > 0, b < 0, 3(a-b)^2 - 2(a+b) < 0 \right\}, \\
E_{34} &= \left\{ (a, b) \in \mathbb{R}^2 : a < 0, b > 0, 3(a-b)^2 - 2(a+b) < 0 \right\}.
\end{aligned} \tag{3.1}$$

Then we clearly see that $E_1 = \bigcup_{i=1}^4 E_{1i}$, $E_2 = \bigcup_{i=1}^7 E_{2i}$, and $E_3 = \bigcup_{i=1}^4 E_{3i}$.

Without loss of generality, we assume that $y > x$. From the symmetry we clearly see that Theorem 1.1 is true if we prove that $M(a, b; x, y) - I(x, y)$ is positive, negative, and neither positive nor negative with respect to $(x, y) \in \{(x, y) \in \mathbb{R}^2 : y > x > 0\}$ for $(a, b) \in E_{11} \cup E_{13}$, $E_{21} \cup E_{23} \cup E_{25} \cup E_{27}$, and $E_{31} \cup E_{33}$.

Let $t = y/x > 1$, then (1.1) and (1.2) lead to

$$\log M(a, b; x, y) - \log I(x, y) = \frac{1}{a+b} \log \frac{t^a + t^b}{2} - \frac{t}{t-1} \log t + 1. \tag{3.2}$$

Let

$$g(t) = \frac{1}{a+b} \log \frac{t^a + t^b}{2} - \frac{t}{t-1} \log t + 1. \tag{3.3}$$

Then simple computations yield

$$\begin{aligned}
\lim_{t \rightarrow 1} g(t) &= 0, \\
g'(t) &= \frac{g_1(t)}{(t-1)^2},
\end{aligned} \tag{3.4}$$

where

$$g_1(t) = \log t - \frac{(t-1)(bt^{b-1} + at^{a-1} + at^b + bt^a)}{(a+b)(t^a + t^b)}. \tag{3.5}$$

Note that

$$g_1(1) = 0, \tag{3.6}$$

$$g'_1(t) = \frac{(t-1)t^{a+b-2}}{(t^a + t^b)^2} f(t), \tag{3.7}$$

where $f(t)$ is defined as in Lemma 2.1.

We divide the proof into three cases.

Case 3. $(a, b) \in E_{11} \cup E_{13}$. We divide our discussion into two subcases.

Subcase 1. $(a, b) \in E_{11}$. From Lemma 2.1(2) we get

$$f(t) > 0 \quad (3.8)$$

for $t > 1$.

Equations (3.3)–(3.8) imply that

$$g(t) > 0 \quad (3.9)$$

for $t > 1$.

Therefore, $M(a, b; x, y) > I(x, y)$ follows from (3.2) and (3.9).

Subcase 2. $(a, b) \in E_{13}$. Then from (1.1), (1.4), and (1.6) together with the monotonicity of the power mean $M_r(x, y)$ with respect to $r \in \mathbb{R}$ for fixed $x, y > 0$, we get

$$M(a, b; x, y) = M(0, b; x, y) = M_b(x, y) \geq M_1(x, y) > I(x, y). \quad (3.10)$$

Case 4. $(a, b) \in E_{21} \cup E_{23} \cup E_{25} \cup E_{27}$. We divide our discussion into four subcases.

Subcase 3. $(a, b) \in E_{21}$. Then Lemma 2.1(1) leads to

$$f(t) < 0 \quad (3.11)$$

for $t > 1$.

Therefore, $M(a, b; x, y) < I(x, y)$ follows from (3.2)–(3.7) and (3.11).

Subcase 4. $(a, b) \in E_{23}$. Then from (1.1), (1.4), and (1.8) together with the monotonicity of the power mean $M_r(x, y)$ with respect to $r \in \mathbb{R}$ for fixed $x, y > 0$ we clearly see that

$$M(a, b; x, y) = M_b(x, y) \leq M_{2/3}(x, y) < I(x, y). \quad (3.12)$$

Subcase 5. $(a, b) \in E_{25}$. Then from Lemma 2.1(3) we know that (3.11) holds again; hence, $M(a, b; x, y) < I(x, y)$.

Subcase 6. $(a, b) \in E_{27}$. Then (1.6) leads to

$$M(a, b; x, y) = M(a, a; x, y) = G(x, y) < I(x, y). \quad (3.13)$$

Case 5. $(a, b) \in E_{31} \cup E_{33}$. We divide our discussion into two subcases.

Subcase 7. $(a, b) \in E_{31}$. Then (2.4) leads to

$$f'(1) > 0. \quad (3.14)$$

Inequality (3.14) and the continuity of $f'(t)$ imply that there exists $\delta_1 > 0$ such that

$$f'(t) > 0 \quad (3.15)$$

for $t \in [1, 1 + \delta_1)$.

From (2.2) and (3.15) we clearly see that

$$f(t) > 0 \quad (3.16)$$

for $t \in (1, 1 + \delta_1)$.

Therefore, $M(a, b; x, y) > I(x, y)$ for $(x, y) \in \{(x, y) \in \mathbb{R}^2 : y > x > 0, y < (1 + \delta_1)x\}$ follows from (3.2)–(3.7) and (3.16).

On the other hand, from (3.3) we clearly see that

$$\lim_{t \rightarrow +\infty} g(t) = -\infty. \quad (3.17)$$

Equations (3.2) and (3.3) together with (3.17) imply that there exists sufficient large $\lambda_1 > 1$ such that $M(a, b; x, y) < I(x, y)$ for $(x, y) \in \{(x, y) \in \mathbb{R}^2 : y > \lambda_1 x > 0\}$.

Subcase 8. $(a, b) \in E_{33}$. Then (2.2) and (2.4) together with the continuity of $f'(t)$ imply that there exists $\delta_2 > 0$ such that

$$f(t) < 0 \quad (3.18)$$

for $t \in (1, 1 + \delta_2)$.

Therefore, $M(a, b; x, y) < I(x, y)$ for $(x, y) \in \{(x, y) \in \mathbb{R}^2 : y > x > 0, y < (1 + \delta_2)x\}$ follows from (3.2)–(3.7) and (3.18).

On the other hand, from (3.3) we clearly see that

$$\lim_{t \rightarrow +\infty} g(t) = +\infty. \quad (3.19)$$

Equations (3.2) and (3.3) together with (3.19) imply that there exists sufficient large $\lambda_2 > 1$ such that $M(a, b; x, y) > I(x, y)$ for $(x, y) \in \{(x, y) \in \mathbb{R}^2 : y > \lambda_2 x > 0\}$. \square

Remark 3.1. Let $E_4 = \{(a, b) \in \mathbb{R}^2 : a + b \neq 0\} \setminus (E_1 \cup E_2 \cup E_3)$, then $E_4 = \{(a, b) \in \mathbb{R}^2 : ab < 0, 3(a - b)^2 - 2(a + b) > 0, 2(a - b)^2 - 3(a + b) + 1 < 0\}$. Unfortunately, in this paper we cannot discuss the case of $(a, b) \in E_4$; we leave it as an open problem to the readers.

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